6 MATRICES

1. Meaning and Definition :

A matrix is an arrangment of numbers in rows and columns. If mn numbers are arranged in a rectangular array of m rows and n columns it is called a matrix of order $m \times n$. In writing a matrix it is usual to enclose this array by big brackets like [ ] or ( ) or { }. The matrix is denoted by capital letters like A, B, C etc. e.g.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots\cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots\cdots & a_{3n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & a_{mn} \end{bmatrix}$$

The individual quantities like $a_{11}, a_{12}, a_{23} \ldots a_{mn}$ are called the elements of the matrix. A matrix of m rows and n columns is said to be of order $m \times n$. It should be made clear at this stage that the matrix is simply an arrangement of numbers and it can not have a value. Matrix should only be looked upon as an operator. The use of matrix algebra is very common in higher Mathematics, Statistics, Economics and Business problems.

Consider the following matrix A

(120)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13}\cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23}\cdots\cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}\cdots\cdots & a_{3n} \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ a_{m_1} & a_{m_2} & a_{m_3} & a_{mn} \end{bmatrix}$$

In this matrix there are m rows and n columns. The matrix is therefore of order $m \times n$. It can be expressed as A_{mxn} All the mn numbers in the matrix are known as elements of the matrix. The element of ith row and j^{th} column is denoted by a_{ij}. In particular,

a_{34} = element of 3rd row and 4th column.

a_{52} = element of 5th row and 2nd column.

a_{44} = element of 4th row and 4th column.

Consider the matrix.

Example:

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 4 & 2 & 3 & 1 \\ 2 & 0 & -5 & -3 \end{bmatrix}$$

Here 12 elements are arranged in 3 rows and 4 columns.

Its order is 3 × 4

In the above matrix

$a_{11} = 1$,	$a_{12} = -2$,	$a_{13} = 3$,	$a_{14} = 0$
$a_{21} = 4$,	$a_{22} = 2$,	$a_{23} = 3$,	$a_{24} = 1$
$a_{31} = 2$,	$a_{32} = 0$,	$a_{33} = -5$,	$a_{34} = -3$

→ **Diagonal elements of a matrix and its principal diagonal**

Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3 rows, 3 columns.
so, order = m × n.
= 3 × 3

This is a matrix of order 3 × 3. The diagonal containing the elements a_{11}, a_{22}, a_{33} is called the principal diagonal of the matrix. a_{11}, a_{22}, a_{33} are diagonal elements.

In the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 5 \\ 2 & 0 & 3 & 4 \\ 5 & 6 & 7 & 2 \\ 4 & 1 & 4 & 3 \end{bmatrix}$$

1, 0, 7, 3 are diagonal elements.

Equal Matrices :

Two matrices are said to be equal if they satisfy the following conditions :

(1) The number of rows in both matrices should be equal and the number of columns should also be equal i.e. the order of both the matrices must be the same.

(2) The corresponding elements in both the matrices should be the same. i.e., two matrices are equal if they are equal in all respects.

Transpose of a matrix :

If we interchange rows and columns of a matrix A, the new matrix so obtained is known as the transpose of matrix A and it is denoted by A^T or A'.

e.g. If $\quad A = \begin{bmatrix} 1 & 2 & 5 & 0 \\ 3 & 5 & 2 & 1 \\ 4 & 1 & 0 & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 1 \\ 5 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The order of A is 3 × 4 and that of A^T is 4 × 3.

i,e., If A is of order $(m × n)$ then A^T is of order $(n × m)$

2. **Special Types of Matrices :**

(i) **Row Matrix :** A matrix in which there is only one row and any number of columns is said to be a row matrix.

$A = [a_{11}, a_{12}, a_{13} ...... a_{1n}]$ is a row matrix.

It is clear that order of this row matrix is 1 × n

e.g., [1 2 3] is a row matrix of order 1 x 3.

(ii) **Column Matrix :** A matrix in which there is only one column and any number of rows is said to be a column matrix.

e.g. $A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ ... \\ ... \\ a_{m1} \end{bmatrix}$ is a column matrix of order m X 1.

Similarly $\begin{bmatrix} 0 \\ 5 \\ 2 \\ 1 \end{bmatrix}$ is a column matrix of order 4×1

(iii) **Zero Matrix or Null Matrix :** If all the elements of matrix are zero, it is said to be a null matrix or a zero matrix.

e.g. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a null matrix of order 2 x 4.

(**Note :** A Zero matrix can be a row matrix or a column matrix).

(iv) **Square Matrix :** A matrix in which number of rows and number of column are equal is said to be a square matrix.

e.g. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3 x 3

Similarly $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is a square matrix of order 2 x 2

(v) **Symmetric Matrix* :** If the transpose of a square matrix gives the same matrix, it is known as a symmetric matrix i.e. for a symmetric matrix,

a_{ij} = element of ith row and j^{th} column

= element of jth row and i^{th} column

= a_{ji}

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* Define symmetric matrix with illustration (Mar. 2015)

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ is a symmetric matrix of order 3×3

Here $\quad a_{21} = a_{12} = 2$

$\qquad a_{13} = a_{31} = 3$

$\qquad a_{23} = a_{32} = 5$

$\therefore \qquad A^T = A$

[Note : Every symmetric matrix must be a square matrix.]

(vi) Skew Symmetric Matrix :* If in a square matrix $a_{ij} = - a_{ji}$ i.e., the elements of ith row and j^{th} column and j^{th} row and i^{th} column are equal in magnitude but opposite in sign, the matrix is known as a skew symmetric marix.

Obviously each diagonal element of a skew symmetric matrix must be zero.

i.e. $(a_{11} = a_{22} = a_{33} = ... = 0)$

e.g. $A = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$ is a skew symmetric matrix of order 3 x 3.

Its diagonal elements are all zero. In a skew symmetric matrix $A^T = - A$.

(vii) Unit Matrix or Identity Matrix : A square matrix in which all diagonal elements are unity and all other elements are zero is known as a unit matrix or an identity matrix, and it is denoted by I.

e.g. $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3×3

Similarly $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2×2

[Note : Unit matrix holds special importance in the study of matrices.]

(viii) Diagonal Matrix : If all elements except diagonal elements of a square matrix are zero the matrix is said to be a diagonal matrix.

e.g. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a diagonal matrix.

* Define skew symmetric matrix with illustration   (Dec. 2015)

3. Simple Rules of Operations on Matrices :

3.1 Addition and Subtraction of Matrices :

The addition or subtraction of two or more matrices is possible only when they are of the same order.

Addition or subtraction of two or more matrices of the same order can be obtained by adding or subtracting the corresponding elements of these matrices. It is obvious that the order of the new matrix so obtained is also same as the order of given matrices.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 2 \end{bmatrix}$ is a matrix of order 2 x 3 and

$B = \begin{bmatrix} -1 & 2 & -1 \\ 3 & -5 & -4 \end{bmatrix}$ is also a matrix of order 2 x 3

$\therefore C = A + B = \begin{bmatrix} 1+(-1) & 2+2 & 3+(-1) \\ 3+3 & 5+(-5) & 2+(-4) \end{bmatrix}$

$\therefore \qquad C = \begin{bmatrix} 0 & 4 & 2 \\ 6 & 0 & -2 \end{bmatrix}$

C is also of order 2 x 3

Similarly

$D = A - B = \begin{bmatrix} 1-(-1) & 2-2 & 3-(-1) \\ 3-3 & 5-(-5) & 2-(-4) \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 & 4 \\ 0 & 10 & 6 \end{bmatrix}$ is of order 2 x 3

3.2 Scalar Product of a Matrix :

The scalar product of a matrix is obtained by multiplying each element of the matrix by that scalar.

e.g. $A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 5 & 7 \end{bmatrix}$ then

$2A = \begin{bmatrix} 2 & 0 & 6 \\ 8 & 10 & 14 \end{bmatrix}$ and

$-3A = \begin{bmatrix} -3 & 0 & -9 \\ -12 & -15 & -21 \end{bmatrix}$

Illustration 1 : Find the elements $a_{11}, a_{23}, a_{13}, a_{34}$ and a_{44} from the following matrices. Give also the order of each matrix.

(1) $\begin{bmatrix} 2 & 3 & 4 & 2 \\ -8 & 7 & -5 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 9 & -3 \\ 1 & 0 & 5 & 6 \\ 6 & 0 & 7 & 9 \end{bmatrix}$

Ans. : (i) In the given matrix there are two rows and four columns.

∴ Its order is 2 x 4.

Here, $a_{11} = 2$, $a_{23} = -5$

$a_{13} = 4$ and the remaining elements are not possible.

(ii) Here the order of the given matrix is 4 x 4.

and $a_{11} = 0$ $a_{23} = 9$ $a_{13} = 1$ $a_{34} = 6$ $a_{44} = 9$.

Illustration 2 : The following matrices are of special type. Mention the type of each one of them.

(1) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & -4 & -5 \\ 2 & -5 & 0 \end{bmatrix}$ (2) $\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ (3) $[0\ 1\ 0\ 1]$

(4) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (5) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (6) $\begin{bmatrix} 0 & -3 & -2 \\ 3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

(7) $\begin{bmatrix} 3 & -4 \\ -3 & 4 \end{bmatrix}$ (8) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Ans. :

(1) It is a symmetric matrix, because by interchanging rows and columns we get the same matrix. (i.e. $A = A^T$)

(2) It is a column matrix, because there is only one column.

(3) It is a row matrix because there is only one row.

(4) It is a null matrix because all the elements are zero.

(5) It is a unit matrix because the elements of principal diagonal are unity and all other elements are zero.

(6) It is a skew symmetric matrix because $a_{ij} = -a_{ji}$ and the elements of principal diagonal are all zero.

(7) It is a square matrix because the number of rows and columns are equal.

(8) It is a diagonal matrix because all the elements except diagonal elements are zero.

Illustration 3 : Find A + B, B – A and A – B from the following matrices :

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 5 & 7 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 0 & -2 \\ 7 & -6 & 0 \end{bmatrix}.$$

Ans. : Here,

Order of A = 2 × 3 and

Order of B = 2 × 3

∴ The addition and subtraction of A and B are possible.

$$A + B = \begin{bmatrix} 0+5 & 1+0 & 2-2 \\ 5+7 & 7-6 & 6+0 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 0 \\ 12 & 1 & 6 \end{bmatrix}$$

$$B - A = \begin{bmatrix} 5-0 & 0-1 & -2-2 \\ 7-5 & -6-7 & 0-6 \end{bmatrix} = \begin{bmatrix} 5 & -1 & -4 \\ 2 & -13 & -6 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 0-5 & 1-0 & 2-(-2) \\ 5-7 & 7-(-6) & 6-0 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 4 \\ -2 & 13 & 6 \end{bmatrix}$$

Illustration 4 : Find A + 2B, A – B + C, A + B – 2C, A – C, A + 2C, B + D from the following matrices (if possible) :

$$A = \begin{bmatrix} 4 & 2 & 0 & 3 \\ 5 & -7 & -2 & 0 \\ 6 & 0 & -3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & -1 & -2 \\ 4 & -2 & 0 & -3 \\ 3 & 0 & 4 & 0 \\ 5 & 3 & 7 & 9 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 2 \\ -3 & 1 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Ans. :

Order of A = 3 × 4

Order of B = 4 × 4

Order of C = 3 × 4

Order of D = 3 × 4.

Thus the orders of A, C and D are equal while the order of B is different.

∴ A + 2B, A − B + C, A + B − 2C and B + D are not possible while.

$$A - C = \begin{bmatrix} 4-1 & 2-0 & 0-0 & 3-2 \\ 5+3 & -7-1 & -2-1 & 0-0 \\ 6-4 & 0-2 & -3-0 & -1-1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 & 1 \\ 8 & -8 & -3 & 0 \\ 2 & -2 & -3 & -2 \end{bmatrix}$$

$$A + 2C = \begin{bmatrix} 4+2 & 2+0 & 0+0 & 3+4 \\ 5-6 & -7+2 & -2+2 & 0+0 \\ 6+8 & 0+4 & -3+0 & -1+2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 & 0 & 7 \\ -1 & -5 & 0 & 0 \\ 14 & 4 & -3 & 1 \end{bmatrix}$$

3.3 Multiplication of Two Matrices :

If $A = \begin{bmatrix} 2 & 5 & 7 \\ 1 & 2 & 3 \end{bmatrix}$ is a matrix of order 2 × 3

and $B = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 7 & 9 \\ 5 & 2 & 4 \end{bmatrix}$ is a matrix of order 3 × 3

Then the multiplication AB can be obtained in the following way : .

$$AB = \begin{bmatrix} \boxed{2 \quad 5 \quad 7} \\ \boxed{1 \quad 2 \quad 3} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 3 \\ 4 \\ 5 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 7 \\ 2 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 9 \\ 4 \end{matrix}} \end{bmatrix}$$

$$AB = \begin{bmatrix} 2\times3+5\times4+7\times5 & 2\times2+5\times7+7\times2 & 2\times1+5\times9+7\times4 \\ 1\times3+2\times4+3\times5 & 1\times2+2\times7+3\times2 & 1\times1+2\times9+3\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 6+20+35 & 4+35+14 & 2+45+28 \\ 3+8+15 & 2+14+6 & 1+18+12 \end{bmatrix}$$

$$= \begin{bmatrix} 61 & 53 & 75 \\ 26 & 22 & 31 \end{bmatrix}$$

The order of matrix AB is 2 x 3.

If A is a matrix of order *m* x *n* and B is a matrix of order *n* x p, then the product AB will be a matrix of order *m* x p. Thus for the multiplication of two matrices A and B, the number of columns of matrix A and the number of rows of matrix B should be equal. For obtaining the first element of the first row of matrix AB, the elements of the first row are multiplied by the corresponding elements of the first column and they are added up. Similarly for obtaining the second element of first row, the elements of the first row of A are multiplied by the corresponding elements of the second column of B and they are added up. For obtaining the third element of the second row of AB, the elements of second row of A are multiplied by the corresponding elements of the third column of B and are added up. Similarly all the elements of the matrix AB can be obtained.

The multiplication of two matrices are explained in the following illustrations :

Illustration 5 :

If $A = \begin{bmatrix} 4 & 5 \\ 3 & 0 \\ 7 & 4 \end{bmatrix}$ and

$B = \begin{bmatrix} 5 & 6 & 7 \\ 1 & 1 & 0 \end{bmatrix}$ find AB.

Ans. : Here A is a matrix of order 3 x 2 and B is a matrix of order 2 x 3 i.e. the number of columns of A is equal to the number of rows of B. Here the product AB is possible and AB will be of order 3 x 3.

Now, $AB = \begin{bmatrix} 4 & 5 \\ 3 & 0 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \\ 1 & 1 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 4 \times 5 + 5 \times 1 & 4 \times 6 + 5 \times 1 & 4 \times 7 + 5 \times 0 \\ 3 \times 5 + 0 \times 1 & 3 \times 6 + 0 \times 1 & 3 \times 7 + 0 \times 0 \\ 7 \times 5 + 4 \times 1 & 7 \times 6 + 4 \times 1 & 7 \times 7 + 4 \times 0 \end{bmatrix}$

$= \begin{bmatrix} 25 & 29 & 28 \\ 15 & 18 & 21 \\ 39 & 46 & 49 \end{bmatrix}$

Illustration 6 :

If $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$

Find AB and BA. Is AB = BA ?

Ans. : Here A is 2 x 2 matrix and B is 2 x 2 matrix. Hence AB is possible.

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 10+0 & 2+9 \\ 5+0 & 1+12 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 11 \\ 5 & 13 \end{bmatrix}$$

B is 2 x 2 matrix and A is 2 x 2 matrix hence BA is also possible.

$$BA = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 10+1 & 15+4 \\ 0+3 & 0+12 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 19 \\ 3 & 12 \end{bmatrix}$$

Thus $AB \neq BA$.

The matrix multiplication is not commutative.

Illustration 7 :

If $A = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 4 & -2 \\ 1 & 0 & -1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 \\ -5 & 2 \\ 4 & 7 \end{bmatrix}$ **find AB. Is BA possible ?**

Ans. : As A is 3 x 3 matrix and B is 3 x 2 matrix, AB is possible.

$$AB = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 4 & -2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -5 & 2 \\ 4 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -2-5+0 & 0+2+0 \\ 3-20-8 & 0+8-14 \\ 1+0-4 & 0+0-7 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 2 \\ -25 & -6 \\ -3 & -7 \end{bmatrix}$$

Now, B is 3 x 2 matrix and A is 3 x 3 matrix, hence BA is not possible. Here number of columns of B is not equal, to number of rows of A.

Illustration 8 :

If $A = \begin{bmatrix} 4 & 5 & 7 \\ -2 & 3 & 0 \end{bmatrix}$ and

$B = \begin{bmatrix} 0 & 1 & 4 & 7 \\ -2 & 1 & 0 & 5 \\ 3 & 0 & 1 & 4 \end{bmatrix}$, find AB.

Ans. : Here A is 2 x 3 matrix and B is 3 x 4 matrix, hence AB is possible.

$$AB = \begin{bmatrix} 4 & 5 & 7 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 & 7 \\ -2 & 1 & 0 & 5 \\ 3 & 0 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0-10+21 & 4+5+0 & 16+0+7 & 28+25+28 \\ 0-6+0 & -2+3+0 & -8+0+0 & -14+15+0 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 9 & 23 & 81 \\ -6 & 1 & -8 & 1 \end{bmatrix}$$

Illustration 9 :

If $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ find the value of $A^2 - A + I$

Ans. : Here A is a square matrix, hence $A^2 = A \times A$ is possible.

$$A^2 = A \times A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+4 & 2+4+4 & 2+4+2 \\ 1+2+4 & 2+4+4 & 2+4+2 \\ 2+2+2 & 4+4+2 & 4+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 10 & 8 \\ 7 & 10 & 8 \\ 6 & 10 & 9 \end{bmatrix}$$

Now, $A^2 - A + I = \begin{bmatrix} 7 & 10 & 8 \\ 7 & 10 & 8 \\ 6 & 10 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 8 & 6 \\ 6 & 9 & 6 \\ 4 & 8 & 9 \end{bmatrix}$$

Illustration 10 : Find the value of : (March, April, 2007, 2009)

$$[1\ 2\ 3] \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 9 \\ 9 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

Ans. : The order of the first matrix is 1 x 3 and that of the second matrix is 3 x 3, hence their product is a matrix of order 1 x 3. If this product matrix is multiplied by the third matrix of order 3 x 1, we get a matrix of order 1 x 1.

$$[1\ 2\ 3] \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 9 \\ 9 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

$$= [2+0+27 \quad 3+8+3 \quad 0+18+0] \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

$$= [29 \ 14 \ 18] \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

$$= [58 + 70 + 18]$$

$$= [146]$$

Illustration 11 : If $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ find matrix B such that

$$A + 2B = A^2$$

Ans. : $A^2 = A \times A$

$$= \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 16 + 2 & 4 + 3 \\ 8 + 6 & 2 + 9 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$$

Now $A + 2B = A^2$

$\therefore \qquad 2B = A^2 - A$

$\therefore \qquad 2B = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 14 & 6 \\ 12 & 8 \end{bmatrix}$$

$\therefore \qquad B = \dfrac{1}{2} \begin{bmatrix} 14 & 6 \\ 12 & 8 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 3 \\ 6 & 4 \end{bmatrix}$$

Illustration 12 :

If $A = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$ Prove that $A^2 = I$

Ans. : $A^2 = A \times A$

$$A^2 = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix} \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0+4-3 & 0-12+12 & 0-12+12 \\ 0-3+3 & 4+9-12 & 3+9-12 \\ 0+4-4 & -4-12+16 & -3-12+16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I.$$

4. Laws of Matrix Algebra :

4.1 Properties of Addition of Matrices :

(1) If A and B be any two matrices of the same order $m \times n$, then
A + B = B + A (Commutative law)

(2) If A, B, C be any three matrices of the same order $m \times n$, then
A + (B + C) = (A + B) + C (Associative law for addition)

(3) If K be a scalar and A, B be two matrices of the same order $m \times n$ then.

$$K (A + B) = KA + KB$$

(4) If A be a $m \times n$ matrix and 0 be a null matrix of the same order, then

(i) A + 0 = 0 + A = A
(ii) A + (− A) = (− A) + A = 0

(5) If A, B, C be any three matrices of the same order $m \times n$, then
A + C = B + C gives A = B.

4.2 Properties of Matrix Multiplication

(1) If A be a square matrix of order $n \times n$ and I be a unit matrix of the same order then. AI = IA = A

(2) If A is a $m \times n$ matrix and 0 is a $n \times m$ matrix, then
A 0 = 0 A = 0

(3) If A, B, C be three matrixes of order $m \times n$, $n \times p$, $p \times q$ respectively, then A(BC) = (AB) C (Associative Law for multiplication)

(4) If A, B, C be three matrices of order $m \times n$, $n \times p$, $n \times p$ respectively, then

A(B + C) = A . B + A . C [Distributive Law]

(5) If A, B, C are three matrices such that AB = AC then, in general B ≠ C

(6) If AB = O where A, B are two matrices, then in general A ≠ O or, B ≠ O, or A ≠ O and B ≠ O.

4.3 Properties of the Transpose of a Matrix

If A^T & B^T be the transposes of two matrices A and B - then

(1) $(A^T)^T = A$

(2) $(A + B)^T = A^T + B^T$

(3) $(AB)^T = B^T \ A^T$

5. Determinant :

Another type of arrangement of numbers is called determinant. In a matrix the number of rows and number of columns are not necessarily equal, but in a determinant number of rows and number of columns should be equal. Thus a determinant is a square arrangement of numbers. A matrix cannot have a value, while the value of a determinants can be found out. The elements of a determinant are shown within two vertical lines.

Thus we have seen two types of arrangements of numbers, i.e. Determinant and Matrix. We shall now understand the difference between the two :

Determinant	Matrix
(1) In determinant the number of rows and columns are equal.	(1) In matrix the number of rows and columns are not necessarily equal.
(2) In determinant the elements are shown between two vertical lines like | |.	(2) In matrix the elements are shown in brackets like ( ), { } or [ ]
(3) A determinant has a value.	(3) A matrix can not have a value. It is merely an arrangement.

Determinant of a Square Matrix :

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ then, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

is called a determinant of matrix A and it is denoted by $|A|$.

Illustration 13 : Find the values of the determinant of the following matrices :

(i) $A = \begin{bmatrix} 2 & 3 \\ 7 & 18 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 5 & 6 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

(iii) $B = \begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$ (iv) $B = \begin{bmatrix} 5 & -1 & 1 \\ -2 & 3 & 4 \\ 1 & 1 & 7 \end{bmatrix}$

Ans. : (i) $|A| = \begin{vmatrix} 2 & 3 \\ 7 & 18 \end{vmatrix} = 36 - 21 = 15$

(ii) $|A| = \begin{vmatrix} 5 & 6 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix}$

$= 5(4 - 3) - 6(0 - 3) + 1(0 - 2)$
$= 5 + 18 - 2$
$= 21$

(iii) $|B| = \begin{vmatrix} -2 & 3 \\ -4 & 5 \end{vmatrix} = -10 + 12 = 2$

(iv) $|B| = \begin{vmatrix} 5 & -1 & 1 \\ -2 & 3 & 4 \\ 1 & 1 & 7 \end{vmatrix}$

$= 5(21 - 4) + 1(-14 - 4) + 1(-2 - 3)$
$= 5(17) + 1(-18) + 1(-5)$
$= 85 - 18 - 5$
$= 62$

Illustration 14 : Find the transpose of the following matrices :

$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 7 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 5 \\ 3 & 4 & 1 \end{bmatrix}$

Ans. :

(i) $A^T = \begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix}$

(ii) $B^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 7 \end{bmatrix}$; $C^T = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 2 & 4 \\ 3 & 5 & 1 \end{bmatrix}$

Adjoint of a Square Matrix :

$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 6 \\ 5 & 7 & 8 \end{bmatrix}$ is a 3 x 3 square matrix. The determinant of this matrix is

$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 0 & 6 \\ 5 & 7 & 8 \end{vmatrix}$

The minor of any element of a matrix can be obtained by eliminating the row and column in which that element lies. If this minor is given proper sign it can be called co-factor of that element. The signs of minors are taken alternatively positive and negative as given below, to obtain co-factors :

$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Thus for the above matrix

minor of 2 $= \begin{vmatrix} 0 & 6 \\ 7 & 8 \end{vmatrix}$ co-factor of 2 $= + \begin{vmatrix} 0 & 6 \\ 7 & 8 \end{vmatrix}$

minor of 3 $= \begin{vmatrix} 1 & 6 \\ 5 & 8 \end{vmatrix}$ co-factor of 3 $= - \begin{vmatrix} 1 & 6 \\ 5 & 8 \end{vmatrix}$

minor of 4 $= \begin{vmatrix} 1 & 0 \\ 5 & 7 \end{vmatrix}$ co-factor of 4 $= + \begin{vmatrix} 1 & 0 \\ 5 & 7 \end{vmatrix}$

minor of 1 $= \begin{vmatrix} 3 & 4 \\ 7 & 8 \end{vmatrix}$ co-factor of 1 $= - \begin{vmatrix} 3 & 4 \\ 7 & 8 \end{vmatrix}$

minor of 0 $= \begin{vmatrix} 2 & 4 \\ 5 & 8 \end{vmatrix}$ co-factor of 0 $= + \begin{vmatrix} 2 & 4 \\ 5 & 8 \end{vmatrix}$

Thus co-factor of each element is found out by giving proper sign to its minor.

co-factor of 6 $= - \begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix}$

co-factor of 5 $= + \begin{vmatrix} 3 & 4 \\ 0 & 6 \end{vmatrix}$

co-factor of 7 $= - \begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix}$

cofactor of 8 = $+\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}$

Adjoint of a square matrix is the transpose of the matrix of the co-factors of a given matrix. In order to obtain adjoint of a matrix first of all in place of each element write its cofactor and then take the transpose of the matrix obtained. The adjoint of a square matrix A is denoted by adj. A. In a 2 X 2 matrix the minors are given signs as $\begin{vmatrix} + & - \\ - & + \end{vmatrix}$ to obtain cofactors.

The method of obtaining adjoint of a matrix is shown in the following illustration :

Illustration 15 : Find the adjoint of the following matrix :

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$$

Ans. : The matrix of the co-factors is,

$$\begin{bmatrix} 4 & -1 \\ -5 & 2 \end{bmatrix}$$

Taking transpose,

$$adj \cdot A = \begin{bmatrix} 4 & -5 \\ -1 & 2 \end{bmatrix}$$

Illustration 16 : Find adj. A

$$A = \begin{bmatrix} 7 & 8 \\ 2 & 10 \end{bmatrix}$$

Ans. : The matrix of the co-factors is,

$$A = \begin{bmatrix} 10 & -2 \\ -8 & 7 \end{bmatrix}$$

Taking transpose,

$$adj. \ A = \begin{bmatrix} 10 & -8 \\ -2 & 7 \end{bmatrix}$$

Illustration 17 : Find adj. of A,

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 5 \\ 3 & 4 & 2 \end{bmatrix}$$

Ans. : The matrix of the co-factors is

$$\begin{bmatrix} \begin{vmatrix} 2 & 5 \\ 4 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -16 & 13 & -2 \\ -2 & 1 & 1 \\ 13 & -9 & 1 \end{bmatrix}$$

Taking transpose,

$$adj \cdot A = \begin{bmatrix} -16 & -2 & 13 \\ 13 & 1 & -9 \\ -2 & 1 & 1 \end{bmatrix}$$

Illustration 18 : Find adj. A and obtain the value of A x (adj. A)

$$A = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 2 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

Ans. : The matrix of the co–factors is

$$\begin{bmatrix} \begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 0 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 7 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 7 \\ 0 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 7 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 7 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -12 & 6 \\ 21 & 6 & -3 \\ -14 & 9 & 2 \end{bmatrix}$$

$$adj \cdot A = \begin{bmatrix} -3 & 21 & -14 \\ -12 & 6 & 9 \\ 6 & -3 & 2 \end{bmatrix}$$

Now, A x (adj•A)

$$= \begin{bmatrix} 1 & 0 & 7 \\ 2 & 2 & 5 \\ 0 & 3 & 6 \end{bmatrix} \begin{bmatrix} -3 & 21 & -14 \\ -12 & 6 & 9 \\ 6 & -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3-0+42 & 21+0-21 & -14+0+14 \\ -6-24+30 & 42+12-15 & -28+18+10 \\ 0-36+36 & 0+18-18 & 0+27+12 \end{bmatrix}$$

$$= \begin{bmatrix} 39 & 0 & 0 \\ 0 & 39 & 0 \\ 0 & 0 & 39 \end{bmatrix}$$

6. Inverse of a Matrix :*

If A is a square matrix and if there exists another square matrix B of the same order such that AB = BA = I then B is called inverse of matrix A and it is denoted by A^{-1}. The necessary condition for a matrix to have an inverse is that its determinant should not be equal to zero. A square matrix whose determinant is not equal to zero, is called a non singular matrix. Thus a non singular square matrix can have an inverse.

To obtain inverse of a Matrix :

If A is a non singular square matrix then its inverse can be obtained in the following way :

$$A^{-1} = \frac{adj\,A}{|A|}$$

Thus for finding inverse of a given matrix, we should first obtain the value of its determinant. If the value is not equal to zero the inverse is possible. We shall now find inverse of some matrices.

Illustration 19 : Find inverse of the following matrices :

(i) $A = \begin{bmatrix} 2 & 3 \\ 4 & 10 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$

Ans. : (i) Here $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 10 \end{vmatrix} = 20 - 12 = 8 \neq 0$

Hence the matrix is non sigular..We shall first of all find adj. A. The matrix of the co-factors is

$$\begin{bmatrix} 10 & -4 \\ -3 & 2 \end{bmatrix}$$

Define inverse of matrix   (Dec. 2014)

Taking transpose

$$adj \cdot A = \begin{bmatrix} 10 & -3 \\ -4 & 2 \end{bmatrix}$$

Now, $A^{-1} = \dfrac{adj \cdot A}{|A|}$

$$= \frac{1}{8}\begin{bmatrix} 10 & -3 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5/4 & -3/8 \\ -1/2 & 1/4 \end{bmatrix}$$

(ii) $\qquad A = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$

Here, $\qquad |A| = \begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} = 10 - 4 = 6 \neq 0$

Thus the matrix is non singular.

We shall now obtain adj. A. The matrix of the co-factors is

$$\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}$$

$$adj \cdot A = \begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

Now, $A^{-1} = \dfrac{adj \cdot A}{|A|}$

$$= \frac{1}{6}\begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & -1/6 \\ -2/3 & 5/6 \end{bmatrix}$$

Illustration 20 : Find A^{-1} and verify that $AA^{-1} = I$.

$$A = \begin{bmatrix} 6 & 3 \\ 4 & 5 \end{bmatrix}$$

Ans. : Here $|A| = \begin{vmatrix} 6 & 3 \\ 4 & 5 \end{vmatrix} = 30 - 12 = 18 \neq 0$

Thus that matrix is non singular.

We shall now obtain adj. A.

The matrix of the co-factors is

$$\begin{bmatrix} 5 & -4 \\ -3 & 6 \end{bmatrix}$$

Taking transpose,

$$adj \cdot A = \begin{bmatrix} 5 & -3 \\ -4 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{adj \cdot A}{|A|}$$

$$= \frac{1}{18} \begin{bmatrix} 5 & -3 \\ -4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{18} & \frac{-1}{6} \\ \frac{-2}{9} & \frac{1}{3} \end{bmatrix}$$

Now, $AA^{-1} = \begin{bmatrix} 6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} \frac{5}{18} & -\frac{1}{6} \\ -\frac{2}{9} & \frac{1}{3} \end{bmatrix}$

$$= \begin{bmatrix} \frac{5}{3} - \frac{6}{9} & -1 + 1 \\ \frac{10}{9} - \frac{10}{9} & -\frac{2}{3} + \frac{5}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

Illustration 21 : Find inverse of the following matrix :

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$
(March/April, 2007, 2009)
2019

Ans. : Here,

$$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(10-6) - 3(0-6) + 1(0-5)$$

$$= 8 + 18 - 5$$

$$= 21 \ (\neq 0)$$

Thus matrix is non-singular.

The matrix of the co-factors is

$$\begin{bmatrix} \begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 6 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 5 \\ 1 & 1 \end{vmatrix} \\[2mm] -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \\[2mm] \begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & -5 \\ -5 & 3 & 1 \\ 13 & -12 & 10 \end{bmatrix}$$

Taking transpose,

$$adj \cdot A = \begin{bmatrix} 4 & -5 & 13 \\ 6 & 3 & -12 \\ -5 & 1 & 10 \end{bmatrix}$$

Now, $\quad A^{-1} = \dfrac{adj \cdot A}{|A|}$

$$= \frac{1}{21} \begin{bmatrix} 4 & -5 & 13 \\ 6 & 3 & -12 \\ -5 & 1 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{4}{21} & -\dfrac{5}{21} & \dfrac{13}{21} \\[2mm] \dfrac{6}{21} & \dfrac{1}{7} & -\dfrac{4}{7} \\[2mm] -\dfrac{5}{21} & \dfrac{1}{21} & \dfrac{10}{21} \end{bmatrix}$$

Illustration 22 : Find the inverse of the following matrix and verify that $AA^{-1} = I$:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

Ans. : Here,

$$|A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(0+1) - 1(2+1) - 1(1-0)$$

$$= 2 - 3 - 1$$

$$= -2 \; (\neq 0)$$

Thus the matrix is non-singular.

The matrix of the co–factors is

$$\begin{bmatrix} \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 1 \\ -3 & 5 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

Taking transpose,

$$adj \cdot A = \begin{bmatrix} 1 & -3 & -1 \\ -3 & 5 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{adj \cdot A}{|A|}$$

$$= -\frac{1}{2}\begin{bmatrix} 1 & -3 & -1 \\ -3 & 5 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

Now, $AA^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix} \times -\frac{1}{2}\begin{bmatrix} 1 & -3 & -1 \\ -3 & 5 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

$$= -\frac{1}{2}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -3 & -1 \\ -3 & 5 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 2-3-1 & -6+5+1 & -2+1+1 \\ 1-0-1 & -3+0+1 & -1+0+1 \\ 1-3+2 & -3+5-2 & -1+1-2 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= I$$

Illustration 23 : Show that $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ Satisfies the

equation $A^3 - 6A^2 + 9A - 4I = 0$
Hence deduce A^{-1}.
Ans. : $\overline{A^2 = A \times A}$

$$= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$A^3 = A^2 \times A$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now, $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix}$$

$$- \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore A^3 - 6A^2 + 9A - 4I = O$

To obtain A^{-1}, we multiply this equation by A^{-1}

$\therefore \quad A^{-1}A^3 - 6A^{-1}A^2 + 9A^{-1}A - 4A^{-1}I = A^{-1}\,0$

$\therefore \quad A^{-1}AA^2 - 6A^{-1}AA + 9I - 4A^{-1} = 0$

$\therefore \quad IA^2 - 6IA + 9I = 4A^{-1}$

$\therefore \quad A^2 - 6A + 9I = 4A^{-1}$

$$\therefore \quad 4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$4A^{-1} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

7. Properties of the Inverse of a Matrix :

If A & B be two square matrices and A^{-1}, B^{-1} are the inverses of A and B respectively then,

(1) $\quad A\,A^{-1} = A^{-1}\,A = I$

(2) $\quad (A^{-1})^{-1} = A$

(3) $\quad (A^T)^{-1} = (A^{-1})^T$

(4) $\quad (AB)^{-1} = B^{-1}\,A^{-1}$

Illustration 24 : A firm has three offices, one each in Ahmedabad, Surat and Rajkot. In Ahmedabad office there is 1 office superintendent, 2 head clerks, 6 clerks and 4 peons. In Surat office there are 2 head clerks, 5 clerks and 3 peons. In Rajkot office there is 1 office superintendent, 1 head clerk, 6 clerks, and 3 peons. The average monthly salary of office superintendent, head clerk, clerk and peon are respectively Rs. 12,000, Rs. 10,000, Rs. 6000 and Rs 3000. Using matrix multiplication find total monthly salary bill of each office.

Ans. : The staff of the office in three different cities can be represented as following matrix :

$$
\begin{array}{c}
\\
\text{Ahmedabad} \\
\text{Surat} \\
\text{Rajkot}
\end{array}
\begin{array}{cccc}
\text{OS} & \text{HC} & \text{C} & \text{P} \\
\left[\begin{array}{cccc}
1 & 2 & 6 & 4 \\
0 & 2 & 5 & 3 \\
1 & 1 & 6 & 3
\end{array}\right]
\end{array}
$$

their monthly salary can be represented as a matrix.

$$
\begin{array}{c}
\text{OS} \\
\text{HC} \\
\text{C} \\
\text{P}
\end{array}
\left[\begin{array}{c}
12,000 \\
10,000 \\
6,000 \\
3,000
\end{array}\right]
$$

The salary bill of the three offices can be obtained by multiplication of two matrices, as shown below :

$$
\begin{array}{c}
\text{Ahmedabad} \\
\text{Surat} \\
\text{Rajkot}
\end{array}
\left[\begin{array}{cccc}
1 & 2 & 6 & 4 \\
0 & 2 & 5 & 3 \\
1 & 1 & 6 & 3
\end{array}\right] \times
\left[\begin{array}{c}
12000 \\
10000 \\
6000 \\
3000
\end{array}\right]
$$

$$
= \left[\begin{array}{cccc}
12000 & +20000 & +36000 & +12000 \\
0 & +20000 & +30000 & +9000 \\
12000 & +10000 & +36000 & +9000
\end{array}\right]
$$

$$
= \left[\begin{array}{c}
80000 \\
59000 \\
67000
\end{array}\right]
$$

∴ Salary bills of Ahmedabad, Surat and Rajkot offices are respectively Rs. 80,000, Rs. 59,000 and Rs. 67,000 per month.

Illustration 25 : A factory produces two types of items A and B. For the production of these two items two machines are used. For producing one unit of item A first machine is used for 2 hours and second machine is used for 5 hours, and producing one unit of item B first machine is used for 3 hours and second machine is used for 1 hour. If the total time available on these two machines are respectively 85 hours and 115 hours, find the number of units of A and B that should be produced.

Ans. :

Suppose x units of A and y units of B are produced.

$$\therefore \quad 2x + 3y = 85$$
$$5x + y = 115$$

Use Cramer's method and get the solution $x = 20$, $y = 15$.

Illustration 26 : If $A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix}$ $B^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

then find $(AB)^{-1}$.

Ans. : We know that $(AB)^{-1} = B^{-1}, A^{-1}$

$$\therefore (AB)^{-1} = B^{-1} A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+1 & 0+0+1 & 0+0+3 \\ 1+6+3 & -1+0+3 & 2+8+9 \\ 1+9+4 & -1+0+4 & 2+12+12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 10 & 2 & 19 \\ 14 & 3 & 26 \end{bmatrix}$$

Illustration 27 : Find the inverse of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

Ans. : Here $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ is the matrix of the co-efficients of the variables x, y, z in the given equations.

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 36 \end{bmatrix} \qquad \text{and}$$

Let, us find first inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} = A$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1(18 - 12) - 1(9 - 3) + 1(4 - 2)$$

$$= 6 - 6 + 2$$
$$= 2 \neq 0$$

Matrix of the co-factors

$$= \begin{bmatrix} 18-12 & -(9-3) & 4-2 \\ -(9-4) & 9-1 & -(4-1) \\ 3-2 & -(3-1) & 2-1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\therefore \text{adj } A = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

Now, $A^{-1} = \dfrac{1}{2} \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & +1 \end{bmatrix}$

Illustration 28 : If $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ $B = \begin{bmatrix} a & b \\ 3 & 5 \end{bmatrix}$ **Find a and b such that** $AB = BA$. **Also compute** $3A + 4B$.

Ans. : $AB = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 3 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 3a+6 & 3b+10 \\ 4a+3 & 4b+5 \end{bmatrix}$$

Now, $BA = \begin{bmatrix} a & b \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 3a+4b & 2a+b \\ 9+20 & 6+5 \end{bmatrix}$$

$$= \begin{bmatrix} 3a+4b & 2a+b \\ 29 & 11 \end{bmatrix}$$

We have AB = BA

$$\therefore \quad \begin{bmatrix} 3a+6 & 3b+10 \\ 4a+3 & 4b+5 \end{bmatrix} = \begin{bmatrix} 3a+4b & 2a+b \\ 29 & 11 \end{bmatrix}$$

$\therefore \quad 3a + 6 = 3a + 4b$ \hspace{2cm} $3b + 10 = 2a + b$

$\quad 4a + 3 = 29 \Rightarrow a = 6.5$ \hspace{1.5cm} $4b + 5 = 11 \Rightarrow b = 1.5$

$$a = 6.5, \ b = 1.5$$

Now, $\quad 3A + 4B = 3\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} + 4\begin{bmatrix} 6.5 & 1.5 \\ 3 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 9 & 6 \\ 12 & 3 \end{bmatrix} + \begin{bmatrix} 26 & 6 \\ 12 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 35 & 12 \\ 24 & 23 \end{bmatrix}$$

EXERCISE

1. Define a matrix and give the difference between matrix and determinant.

2. Explain with illustrations different types of matrices.

3. Define the following matrices, matrix, unit matrix, Null matrix, Transpose of a matrix, Adjoint of a matrix

4. Define inverse of a matrix and give the condition for the existence of the inverse.

5. Give the condition for addition of two matrices and multiplication of two matrices.

6. The following matrices are of special types. Indicate their types and also give their orders.

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$, \hspace{0.5cm} (ii) $\begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, \hspace{0.5cm} (iii) $\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 4 \\ -2 & -4 & 0 \end{bmatrix}$

(iv) $[6 \ 0 \ 1]$, \hspace{0.5cm} (v) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, \hspace{0.5cm} (vi) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(vii) $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}$, (viii) $\begin{bmatrix} 4 & 0 \\ 7 & 6 \end{bmatrix}$

(ix) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,   (x) $\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$

[Ans. :   (i) Square symmetric of order 3×3

(ii) Column matrix of 4×1

(iii) Skew symmetric of order 3×3

(iv) Row matrix of 1×3

(v) Null matrix, 2×2

(vi) Unit matrix, 3×3

(vii) Diagonal matrix, 3×3

(viii) Square matrix of 2×2

(ix) Unit matrix of 2×2

(x) Skew symmetric of order 2×2]

7.   Find $A + 4B$; $A - 3C$; $B + C - 2D$; $A + 2D$ $A + B - 4C$
(Whichever is possible)

$$A = \begin{bmatrix} 1 & 5 & 2 & 7 & 0 \\ 5 & 2 & 4 & 6 & -3 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 6 & 7 \\ 8 & 9 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 3 & -1 \\ 4 & 6 \end{bmatrix}; \quad D = \begin{bmatrix} 5 & 4 \\ -3 & 2 \\ 5 & -3 \\ -1 & 0 \end{bmatrix}$$

[Ans. :

B,C,D are of orders 4×2 and A is of order 2×5
∴ $A + 4B$; $A - 3C$; $A + 2D$; $A + B - 4C$ are not possible.

$$\text{Whereas} \quad B + C - 2D = \begin{bmatrix} -9 & -5 \\ 12 & 1 \\ -1 & 12 \\ 14 & 15 \end{bmatrix}$$

8.   If $A = \begin{bmatrix} 7 & 3 & -5 \\ 0 & 4 & 2 \\ 1 & 5 & 4 \end{bmatrix}$; and $B = 3A$; $C = B + 2A - 5I$ find the

matrix D such that $D = 2A + B - C$.

[Ans. : D = 5I]

9. If $A = \begin{bmatrix} 6 & 3 \\ -3 & 9 \\ 12 & -6 \end{bmatrix}$, find the matrix B such that $2A^T + 3B = O$

[Ans. : $B = \begin{bmatrix} -4 & 2 & -8 \\ -2 & -6 & 4 \end{bmatrix}$]

10. Find $A + B - C$ and mention the type of the matrix obtained.

$A = \begin{bmatrix} 2 & 5 & 7 \\ 6 & 2 & 4 \\ 0 & 1 & 6 \end{bmatrix}$; $B = \begin{bmatrix} 0 & -4 & -7 \\ -3 & 1 & -5 \\ 1 & -1 & -4 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

[Ans. : Unit matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$]

11. Find $2A + B - C$ and mention the type of the matrix obtained.

$A = \begin{bmatrix} 1 & 4 & 6 \\ 3 & 6 & 4 \\ 2 & 1 & 9 \end{bmatrix}$; $B = \begin{bmatrix} 5 & 2 & 2 \\ 4 & 3 & 3 \\ 3 & 2 & 1 \end{bmatrix}$;

$C = \begin{bmatrix} 6 & 10 & 14 \\ 10 & 14 & 11 \\ 7 & 4 & 18 \end{bmatrix}$

[Ans. : Unit matrix]

12. Find AB and BA.

$A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

[Ans. : $AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $BA = \begin{bmatrix} -11 & 6 & -1 \\ -22 & 12 & -2 \\ -11 & 6 & -1 \end{bmatrix}$

13. If $A = \begin{bmatrix} 2 & 6 \\ 7 & 2 \end{bmatrix}$; $B = \begin{bmatrix} -3 & 5 \\ 0 & 8 \end{bmatrix}$; $C = \begin{bmatrix} 4 & 7 \\ 9 & 5 \end{bmatrix}$

Prove that $A(BC) = (AB)C$ (March, April, 2007)

14. If $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$; $B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

$$C = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Prove that   (i) AB = BA = 0

               (ii) AC = A and (iii) CA = C

15. If $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$

Find AB.

[Ans. : $\begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$]

16. If $A = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$ prove that $A^2 = I$

2019 17. If $A = \begin{bmatrix} 4 & 5 \\ 6 & -2 \\ 3 & 7 \end{bmatrix}$; $B = \begin{bmatrix} 2 & 5 & 7 \\ 8 & 4 & -3 \end{bmatrix}$ find AB and BA.

[Ans. : $AB = \begin{bmatrix} 48 & 40 & 13 \\ -4 & 22 & 48 \\ 62 & 43 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 59 & 49 \\ 47 & 11 \end{bmatrix}$]

18. Prove that

$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ statisfies the following equation :

$A^2 - 4A - 5I = O$

Where O is a null matrix and I is a unit matrix.

19. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ prove that

$A^3 - 3A^2 - A + 9I = O$, Where O is a null matrix of order 3 x 3.

20. **(a)** If $A = [1, 2, 3]$ and $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ find AB and BB'.

(b) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ find value of $A^2 + 5A + 2I$

[Ans. : (a) [32]; $\begin{bmatrix} 16 & 20 & 24 \\ 20 & 25 & 30 \\ 24 & 30 & 36 \end{bmatrix}$; (b) $\begin{bmatrix} 14 & 20 \\ 30 & 44 \end{bmatrix}$]

21. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$ prove that

$$(A + B)^2 = A^2 + B^2$$

22. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & x \\ 4 & y \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, find x and y.

[Ans. : $x = 1$, $y = -1$,]

23. Find the transpose of :

$$\begin{bmatrix} 5 & 6 & -7 & 5 & 0 \\ 4 & 3 & 0 & 1 & 2 \\ -6 & 2 & 1 & -3 & -4 \end{bmatrix}$$

[Ans. : $\begin{bmatrix} 5 & 4 & -6 \\ 6 & 3 & 2 \\ -7 & 0 & 1 \\ 5 & 1 & -3 \\ 0 & 2 & -4 \end{bmatrix}$]

24. If $A = \begin{bmatrix} 2 & 5 & 7 \\ 2 & -1 & 0 \\ 3 & 4 & 8 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 4 & 9 \\ 3 & -2 & 4 \\ -5 & 6 & 8 \end{bmatrix}$

Verify that   (i) $(A + B)^T = A^T + B^T$

(ii) $(AB)^T = B^T A^T$

25. Find adjoint of the following matrices :

(i) $\begin{bmatrix} 2 & -5 \\ 7 & 9 \end{bmatrix}$;   (ii) $\begin{bmatrix} -4 & 3 \\ 8 & 7 \end{bmatrix}$

[Ans. : (i) $\begin{bmatrix} 9 & 5 \\ -7 & 2 \end{bmatrix}$; (ii) $\begin{bmatrix} 7 & -3 \\ -8 & -4 \end{bmatrix}$]

26. Find adjoint of the following matrices :

(i) $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$; (ii) $\begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}$

[Ans. : (i) $\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$; (ii) $\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$]

27. Find inverse of the following matrices :

(i) $\begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix}$

[Ans. : (i) $\begin{bmatrix} 7/29 & 3/29 \\ 5/29 & -2/29 \end{bmatrix}$ (ii) $\begin{bmatrix} -1/10 & 3/10 \\ -4/10 & 2/10 \end{bmatrix}$]

28. Find inverse of the following matrix :

$\begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 1 \\ 4 & -5 & 2 \end{bmatrix}$ [Ans. : $\begin{bmatrix} -\frac{7}{5} & \frac{11}{5} & 1 \\ -\frac{8}{5} & \frac{14}{5} & 1 \\ -\frac{6}{5} & \frac{13}{5} & 1 \end{bmatrix}$]

29. Find inverse of the following matrix :

$A = \begin{bmatrix} 10 & 15 & 20 \\ 20 & 15 & 5 \\ 5 & 10 & 20 \end{bmatrix}$ [Ans. : $\frac{1}{25} \begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$]

30. Examine whether the following matrices are non-singular or not. Find also the inverse of a non-singular matrix.

(i) $\begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

[Ans. :   (i) The matrix is singular. Hence inverse cannot exist.

(ii) The matrix is non-singular. Its inverse is

$\frac{-1}{19} \begin{bmatrix} -1 & 6 & -1 \\ 5 & 8 & -14 \\ -3 & -1 & -3 \end{bmatrix}$]

31. Find the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

[Ans. : $\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$]

32. Find the inverse of

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$[\text{Ans.}: \begin{bmatrix} -3/11 & 4/11 & 5/11 \\ 9/11 & -1/11 & -4/11 \\ 5/11 & -3/11 & -1/11 \end{bmatrix}]$$

33. Find the inverse of

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -3 \\ -1 & 1 & 2 \end{bmatrix} \qquad [\text{Ans.}: \frac{1}{16}\begin{bmatrix} 5 & 7 & 3 \\ -1 & 5 & 9 \\ 3 & 1 & 5 \end{bmatrix}]$$

34. If $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$, find $A + A^T + A^{-1}$

$$[\text{Ans.}: \begin{bmatrix} 11 & 2 \\ 7 & 13 \end{bmatrix}]$$

35. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix}$; $B = \begin{bmatrix} -\frac{13}{4} & \frac{5}{2} & -\frac{1}{4} \\ \frac{5}{2} & -2 & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$

Show that B is the inverse of A.

36. If $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 5 & 6 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ Show that $(AB)^T = B^T A^T$

37. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ Show that $(AB)^{-1} = B^{-1} \cdot A^{-1}$.

38. If $A = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$ and $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find matrix B.

$$[\text{Ans.}: B = A^{-1} = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}]$$

39. If $A = \begin{bmatrix} 5 & 3 & 2 \\ -4 & 2 & 3 \\ 3 & 8 & 1 \end{bmatrix}$; $H = \begin{bmatrix} 6 & -2 & 7 \\ 3 & -1 & -6 \\ 0 & 9 & -1 \end{bmatrix}$ find $(2H - I)$ A

where I is a unit matrix

$$[\text{Ans.}: \begin{bmatrix} 113 & 137 & 24 \\ 6 & -84 & -9 \\ -81 & 12 & 51 \end{bmatrix}]$$

40. A manufacturer produces four products A, B, C and D. These products are sold in two shops. The monthly sale in the two shops are given below :

	Products			
	A	B	C	D
Shop I	800	600	400	100
II	600	500	400	200

If the selling price of the products are respectively Rs 10, 20, 30 and 40, find the total monthly revenue of each shop, using matrix multiplication.

[Ans. : Shop I Rs. 36000 ; Shop II Rs. 36000]

41. In one family A, there are 2 men, 2 women and 3 children. In other family B there are 3 men, 4 women and 3 children. The daily requirements for a man, a woman and a child are respectively 2500, 2000 and 1600 of calories and 60 gms, 50 gms, and 35 gms of proteins. Find total requirements of calories and proteins in each family by using matrix multiplication.

$$\text{[Ans. : } \begin{array}{c} A \\ B \end{array} \begin{bmatrix} 13800 & 325 \\ 20300 & 485 \end{bmatrix} \text{]}$$

42. A factory produces two items X and Y. These items are produced on two machines M_1 and M_2. For producing each unit of X, 2 hours of machine M_1 and 5 hours of machine M_2 are used and for producing each unit of Y, 3 hours of machine M_1 and 2 hours of machine M_2 are used. The total hours available per week on machine M_1 is 50 hours and on machine M_2 is 70 hours. Find how many units of X and Y should be produced.

[Ans. : 10 units of X and 10 units of Y]

43. A person buys 2 pineapples, 3 mangoes and 4 apples in Rs. 43. Another person buys 1 pineapple, 4 mangoes and 2 apples in Rs. 34. And a third person buys 5 pineapples, 2 mangoes and 3 apples in Rs 66. Find the price of each fruit using inverse of a matrix.

[Ans. : Rs. 10 for eaeh pineapple,

Rs. 5 for each mango,

Rs. 2 for each apple]

44. If $U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, find matrix A such that $A = aI_3 + bU + cU^2$

$$\text{[Ans. : } A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix} \text{]}$$

45. If $\begin{bmatrix} 4 & 10 \\ 3 & 9 \end{bmatrix} X = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$ then find X.

46. If $A = \begin{bmatrix} o & a & b \\ -a & o & c \\ -b & -c & o \end{bmatrix}$ and $a^2 + b^2 + c^2 = 1$

prove that $A^3 + A = 0$.

47. If $A = \begin{bmatrix} 5 & -3 & 0 \\ 3 & 2 & 1 \\ 3 & 8 & 7 \end{bmatrix}$ $B = \begin{bmatrix} -8 & 5 & 3 \\ 2 & 7 & -9 \\ 8 & 3 & 0 \end{bmatrix}$

verify that (i) $(A + B)^T = A^T + B^T$
(ii) $(AB)^T = B^T \ A^T$

48. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$; $B = \begin{bmatrix} -1 & 4 \\ 5 & 7 \end{bmatrix}$ then verify that

$(A + B)^2 = A^2 + 2AB + B^2$.

49. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ verify that (AB)

$C = A(BC)$.

50. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then prove that $(aI + bE)^3$

$= a^3I + 3a^2bE$.

University Questions

(April, 2010)

1. **Answer the following :**
(1) Define skew symmetric matrix

(2) If $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -1 \\ 3 & 0 \end{bmatrix}$ then find

$A - 2 B - I.$ [Ans. : $A - 2B - I = \begin{bmatrix} 4 & 6 \\ -4 & 6 \end{bmatrix}$]

(3) Find the inverse of matrix

$A = \begin{bmatrix} 5 & -3 & 0 \\ 3 & 2 & 1 \\ 3 & 8 & 7 \end{bmatrix}$ [Ans. : $A^{-1} = \begin{bmatrix} \dfrac{1}{14} & \dfrac{1}{4} & \dfrac{-1}{28} \\ \dfrac{-3}{14} & \dfrac{5}{12} & \dfrac{-1}{42} \\ \dfrac{3}{14} & \dfrac{-7}{12} & \dfrac{19}{84} \end{bmatrix}$]

(4) If $A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix}$ then verify that

$(A + B)^2 = A^2 + 2AB = B^2$

2019 (5) If $A = \begin{bmatrix} 4 & 5 \\ -1 & 8 \end{bmatrix}$ then find $A + A^T + A^{-1} + I$

[Ans. : $\begin{bmatrix} \dfrac{341}{37} & \dfrac{143}{37} \\ \dfrac{149}{37} & \dfrac{633}{37} \end{bmatrix}$]

2. **Answer the following**   *(March/April 2009)*

(1) If $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$ then find minors and cofactors of elements of A.

[Ans. : Matrix of minors $= \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$. Matrix of cofactors $=$

$\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$]

(2) Define inverse of matrix and find inverse of the matrix

$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \\ 1 & 1 & 2 \end{bmatrix}$ *(Also in March/April 2007)*

[Ans. : $A^{-1} = \begin{bmatrix} \dfrac{4}{21} & \dfrac{-5}{21} & \dfrac{13}{21} \\ \dfrac{6}{21} & \dfrac{1}{7} & \dfrac{-4}{7} \\ \dfrac{-5}{21} & \dfrac{1}{21} & \dfrac{10}{21} \end{bmatrix}$]

(3) Find $(AB)^{-1}$ If $A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & -3 & 0 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 6 & 3 & 1 \\ 2 & 4 & -8 \\ 3 & -6 & 1 \end{bmatrix}$

(Oct./Nov. 2009)

[Ans. : $\begin{bmatrix} 24 & 24 & 36 \\ -30 & 48 & 30 \\ -15 & -27 & -27 \end{bmatrix}$]

3. **Answer the following.**

(1) Define Symmetric matrix with illustration.

(2) If $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ then find adj. A.

[Ans. : Adj. $A = \begin{bmatrix} -4 & -2 \\ -3 & -1 \end{bmatrix}$]

(3) Prove that $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies $A^2 - 4A - 5I = 0$.

(4) If $A = \begin{bmatrix} 5 & -4 & 0 \\ 2 & -2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 2 & 3 \\ 8 & 5 & 0 \\ 1 & 1 & -2 \end{bmatrix}$ then verify that

(i) $(A + B)^T = A^T + B^T$
(ii) $(A^T)^T = A$.

(5) If $A = \begin{bmatrix} -1 & 4 \\ 3 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 5 \\ 2 & 9 \end{bmatrix}$ then show that $(A.B)^{-1} = B^{-1} . A^{-1}$.

4. **Answer the following.**   *(Sept./Oct. 2009)*

(1) Find adjoint of the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

[Ans. :adj. $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$]

(2) Define transpose of a matrix with illustration.

(3) Find the inverse of the matrix

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -8 \\ 6 & -3 & 0 \end{bmatrix}$ [Ans. : $A^{-1} = \begin{bmatrix} \dfrac{-4}{27} & \dfrac{1}{18} & \dfrac{14}{8} \\ \dfrac{-8}{27} & \dfrac{1}{9} & \dfrac{-7}{81} \\ \dfrac{5}{27} & \dfrac{-7}{54} & 0 \end{bmatrix}$]

(4) If $A = \begin{bmatrix} 4 & 5 & 7 \\ -2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 & 0 \\ -2 & 1 & 0 & 2 \\ 0 & 1 & 4 & 7 \end{bmatrix}$ then find AB.

[Ans. : $AB = \begin{bmatrix} -6 & 20 & 40 & 59 \\ -8 & -1 & -6 & 6 \end{bmatrix}$]

(5) Find A^{-1} and verify that $A.A^{-1} = I$ for $A = \begin{bmatrix} 6 & 3 \\ 4 & 5 \end{bmatrix}$.

$$\left[\text{Hint} : A^{-1} = \begin{bmatrix} \dfrac{5}{18} & -\dfrac{1}{6} \\ -\dfrac{2}{9} & \dfrac{1}{3} \end{bmatrix}\right]$$

(6) If $A = \begin{bmatrix} 2 & 5 & 7 \\ 2 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 & 8 \\ 9 & -10 & 11 \\ 13 & 14 & -15 \end{bmatrix}$

Then verify that $(A + B)^T = A^T + B^T$.

5. Answer the following : *(Nov./Dec. 2008)*

(1) What is the necessary condiction for the multiplication of two matrices ?

(2) Find inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

6. Answer the following : *(March/April 2007)*

(1) If $A = \begin{bmatrix} 0 & 1 & 2 \\ 5 & 7 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 & -2 \\ 7 & -6 & 0 \end{bmatrix}$

then find $A + B$ and $B - A$. [Ans. : $A + B = \begin{bmatrix} 5 & 1 & 0 \\ 12 & 1 & 6 \end{bmatrix}$

and $B - A = \begin{bmatrix} 5 - 1 - 4 \\ 2 - 13 - 6 \end{bmatrix}$]

(2) If $A = \begin{bmatrix} 2 & 6 \\ 7 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 5 \\ 0 & 8 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 7 \\ 9 & 5 \end{bmatrix}$, Prove that $A(BC) = (AB)\,C$

(3) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ find the value of $A^2 - A + I$.

(Dec. 2007) [Ans.: $\begin{bmatrix} 6 & 8 & 6 \\ 6 & 9 & 6 \\ 4 & 8 & 9 \end{bmatrix}$]

7. **Answer the following :**

(1) If $A = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 \\ 3 & \frac{1}{2} \end{bmatrix}$ then find $A - 2B$ and

$A + B$.

[Ans. : $A - 2B = \begin{bmatrix} 2 & 7 \\ -5 & 1 \end{bmatrix}$, $A + B = \begin{bmatrix} -1 & 1 \\ 4 & 2\frac{1}{2} \end{bmatrix}$]

(2) Find the inverse of $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 1 \\ 4 & -5 & 2 \end{bmatrix}$

[Ans. : $A^{-1} = \begin{bmatrix} \frac{-7}{5} & \frac{11}{5} & 1 \\ \frac{-8}{5} & \frac{14}{5} & 1 \\ \frac{-6}{5} & \frac{13}{5} & 1 \end{bmatrix}$]

(3) If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ then obtain $A^2 - 5A + 3I$.

[Ans. : $A^2 - 5A + 3I = \begin{bmatrix} 2 & -7 & -5 \\ -3 & 0 & 3 \\ 9 & 9 & 3 \end{bmatrix}$]

(4) Find the inverse of $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ -1 & -1 & -1 \end{bmatrix}$

[$\because |A| = 0$, A^{-1} does not exist.]

8. **Answer the following :**   [March 2016]

(1) Find the value of $\begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix}$

(2) If $A = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$ then find that A^2

(3) Show that $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ satisfy the equation $A^3 - 6A^2$

$+ 9A - 4I = 0$

(4) If $A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix}$ $B^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ then find $(A \cdot B)^{-1}$

(5) Solve the following system of equations using crammer's rule
$ax + by - ab = 0$
$bx + ay - ab = 0$

(6) Solve the following system of equations using crammers rule
$x + 2y - z = 2$
$3x + 6y + z = 1$
$3x + 3y + 2z = 3$

9. **Answer the following :**   [Dec. 2015]
(1) Define skew symmetric matrix with illustration

(2) $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & -5 \\ 2 & 0 & 4 \end{bmatrix}$ then find A^{-1}. Also verify that $A^{-1} \cdot A = 1$

(3) If $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

then Prove that
(i) A.B = B.A. = 0
(ii) A. C. = A
(iii) C. A = C

(4) If $A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & -3 & 0 \end{bmatrix}$ $B^{-1} = \begin{bmatrix} 6 & 3 & 1 \\ 2 & 4 & -8 \\ 3 & -6 & 1 \end{bmatrix}$ then find $(B.A)^{-1}$

(5) Solve the following system of equations using crammer's rule

$x + 6y = 2xy$

$3x + 2y = 2xy$

(6) Solve the following system of equations using crammer's rule

$x + 2y = 3$

$y - 3z = 4$

$3x - 2z = 5$

■ ■ ■

DETERMINANTS & CRAMER'S RULE

1. Determinants of Second Order	4. Use of Determinants in Solving Simultaneous Equations (CRAMER'S METHOD)
2. Determinants of Third Order	(A) For two linear equations
3. Rules of Determinants	(B) For three linear equations
	5. Exercise.

: SYLLABUS :

Determinants, Cramer's Rule

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1. Determinant of Second Order :

Determinant is a system of representing numbers in a square arrangement under a particular symbol.

$ad-bc$ can be represented in the from $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

This representation is called a determinant and $ad - bc$ is the value of the determinant.

$$\therefore \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

In this determinant there are two rows and two columns. Hence it is called a determinant of 2nd order. *a b c* and *d* are known as the elements of the determinant. The first element in the first row is the leading element of the determinant and the diagonal passing through it is the leading diagonal of the determinant.

We shall find the values of some determinants of 2nd order.

(i) $\begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} = (3 \times 5) - (1 \times 4) = 15 - 4 = 11$

(ii) $\begin{vmatrix} 7 & 8 \\ 3 & 2 \end{vmatrix} = (7 \times 2) - (8 \times 3) = -10$

(iii) $\begin{vmatrix} 5 & 20 \\ 3 & 12 \end{vmatrix} = (5 \times 12) - (20 \times 3) = 0$

(iv) $\begin{vmatrix} p & q \\ a & b \end{vmatrix} = pb - aq$

(v) $\begin{vmatrix} x+y & x \\ x & x-y \end{vmatrix} = (x + y)(x - y) - x^2$

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$$= x^2 - y^2 - x^2 = -y^2$$

(vi) $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix} = (a+b)(a+b) - (a-b)(a-b)$

$$= (a+b)^2 - (a-b)^2$$
$$= a^2 + 2ab + b^2 - (a^2 - 2ab + b^2)$$
$$= a^2 + 2ab + b^2 - a^2 + 2ab - b^2$$
$$= 4ab$$

2. Determinant of Third Order :

Determinant of 3rd order consists of three rows and three columns with 9 elements.

e. g $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

In this determinant 9 elements are arranged in 3 rows and 3 columns. Similarly

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is a determinant of 3rd order.

In this determinant a_1 is the leading element and the diagonal passing through it is the leading diagonal. The value of this determinant can be obtained by expanding it. Before we expand the determinant of 3rd order, it is necessary to know the meaning of minor of any element of a determinant.

Minor :

If the column and row passing through an element of a determinant are deleted, then the determinant obtained from the remaining elements is known as the minor of that particular element. It is obvious that minor of any element of a determinant of 3rd order will be a determinant of 2nd order.

In the above determinant of 3rd order.

minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

minor of b_1 is $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

minor of c_1 is $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

Thus we, can have 9 minors for 9 elements of a 3 × 3 determinant. We can expand a given determinant with respect to any row or any column. Take the products of every element of a row or a column with its minor. Observe the order of that particular row or column. If the order is odd give the signs +, −, + respectively to the products and if the order is even give the signs −, +, − respectively to the products. Taking the summation of the products obtained in this way we can get the value of the determinant. e.g. Let us expand.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ with respect to its first row.

The value of the determinant

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

(As the order of row is odd)

$$= a_1(b_2 c_3 - c_2 b_3) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$$
$$= a_1 b_2 c_3 - a_1 c_2 b_3 - b_1 a_2 c_3 + b_1 a_3 c_2 + c_1 a_2 b_3 - c_1 a_3 b_2$$
$$= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1$$

Now let us expand the determinant with respect to second column. The value of the determinant

$$= - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

(As the order of the column is even)

$$= - b_1(a_2 c_3 - a_3 c_2) + b_2(a_1 c_3 - a_3 c_1) - b_3(a_1 c_2 - a_2 c_1)$$
$$= - a_2 b_1 c_3 + a_3 b_1 c_2 + a_1 b_2 c_3 - a_3 b_2 c_1 - a_1 b_3 c_2 + a_2 b_3 c_1$$
$$= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1$$

Thus the value of the determinant is the same, when it is expanded with respect to any row or any column.

We shall now obtain the values of some determinants.

Illustration 1 : Expand the following determinant :

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Ans. : Expanding the determinant with respect to first row, we get the value of the determinant.

$$D = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1 (45 - 48) - 2(36 - 42) + 3(32 - 35)$$
$$= -3 + 12 - 9$$
$$= 0$$

Illustration 2 : Expand the determinant :

$$\begin{vmatrix} 4 & 7 & 8 \\ 3 & 2 & 6 \\ 1 & 5 & 0 \end{vmatrix}$$

Ans. :
$$D = 4(2 \times 0 - 6 \times 5) - 7(3 \times 0 - 1 \times 6) + 8(3 \times 5 - 2 \times 1)$$
$$= 4(-30) - 7(-6) + 8(13)$$
$$= -120 + 42 + 104$$
$$= 26$$

Illustration 3 : Find the value of :

$$\begin{vmatrix} 2 & 5 & -3 \\ 3 & -2 & 5 \\ -1 & 3 & 2 \end{vmatrix}$$

Ans. : Expanding with respect to first row
$$= 2(-4 - 15) - 5(6 + 5) - 3(9 - 2)$$
$$= 2(-19) - 5(11) - 3(7)$$
$$= -38 - 55 - 21$$
$$= -114$$

Illustration 4 : Find the value of

$$\begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix}$$

Ans. :

$$\begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix}$$

$$= 1\left\{ca \times \frac{1}{c} - ab \times \frac{1}{b}\right\} - 1\left\{bc \times \frac{1}{c} - ab \times \frac{1}{a}\right\}$$
$$+ 1\left\{bc \times \frac{1}{b} - ca \times \frac{1}{a}\right\}$$
$$= 1(a - a) - 1(b - b) + 1(c - c)$$
$$= 0 - 0 + 0$$
$$= 0$$

3. Rules of Determinant :

Let

$$D = \begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$$

Expanding w.r.t. first row,

$$D = 4(6-6) - 5(3-3) + 6(2-2)$$
$$= 0.$$

We observe that in the above determinant the elements of two rows are identical.

Fot this we can give the following rule.

Rule I :

(If the elements of any two rows or two columns of a determinant are dentical, the value of the determinant is zero.)

Now let us find the values of the following two determinants :

$$D_1 = \begin{vmatrix} 3 & 2 & 5 \\ 1 & 4 & 10 \\ 2 & 3 & 15 \end{vmatrix} \text{ and } D_2 = 5 \times \begin{vmatrix} 3 & 2 & 1 \\ 1 & 4 & 2 \\ 2 & 3 & 3 \end{vmatrix}$$

Expanding D_1

$$\begin{aligned} D_1 &= 3(60-30) - 2(15-20) + 5(3-8) \\ &= 90 + 10 - 25 \\ &= 75 \end{aligned}$$

Similarly

$$\begin{aligned} D_2 &= 5[3(12-6) - 2(3-4) + 1(3-8)] \\ &= 5[18 + 2 - 5] \\ &= 5 \times 15 \\ &= 75 \end{aligned}$$

It is seen that in the first determinant from each element of third column can be taken as the common factor. This factor is taken common in the second determinant, however the values of both determinants are the same. Thus, we have the following rule :

Rule II :

If there is a common factor in each element of any row or any column of a delerminant it can be taken as the common factor of the leterminant.

Let us expand the determinant.

$$\begin{vmatrix} 3 & 1 & 3 \\ 5 & 2 & 2 \\ 7 & 4 & 8 \end{vmatrix}$$

$= 3(16 - 8) - 1(40 - 14) + 3(20 - 14)$

$= 24 - 26 + 18$

$= 16$

Interchanging the second and third columns. We get the determinant

$$\begin{vmatrix} 3 & 3 & 1 \\ 5 & 2 & 2 \\ 7 & 8 & 4 \end{vmatrix}$$

Its value $= 3(8 - 16) - 3(20 - 14) + 1(40 - 14)$

$= - 24 - 18 + 26)$

$= - 16$

Thus only the sign of the value of the determinant is changed by interchanging two columns. The rule for this can be given as follows :

Rule III :

If two rows or columns of a determinant are interchanged the value of the determinant is changed only in sign.

Consider the determinant $\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$

Its value $= 4(4 - 3) - 5(2 - 9) + 6(1 - 6)$

$= 4 + 35 - 30$

$= 9.$

Now multiplying each element of the second row by 2 and adding them to the corresponding elements·of third row, we get the new determinant as

$$\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 5 & 5 & 8 \end{vmatrix}$$

Its value $= 4(16 - 15) - 5(8 - 15) + 6(5 - 10)$

$= 4 + 35 - 30$

$= 9.$

Hence the value of the two determinants are the same. We can write the following rule from this.

Rule IV :

If all the elements of any row (or column) are multiplied by a constant number and added to the corresponding elements of any other

row (or column) the value cf the determinant remains unchanged.

Now we shall show that

$$\begin{vmatrix} 3+2 & 2+3 & 1+3 \\ 5 & 3 & 1 \\ 2 & 2 & 8 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 1 \\ 5 & 3 & 1 \\ 2 & 2 & 8 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 3 \\ 5 & 3 & 1 \\ 2 & 2 & 8 \end{vmatrix}$$

Here

$$\text{L.H.S.} = \begin{vmatrix} 3+2 & 2+3 & 1+3 \\ 5 & 3 & 1 \\ 2 & 2 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 5 & 4 \\ 5 & 3 & 1 \\ 2 & 2 & 8 \end{vmatrix}$$

$$= 5(24 - 2) - 5(40 - 2) + 4(10 - 6)$$

$$= 110 - 190 + 16$$

$$= -64.$$

$$\text{R.H.S.} = \begin{vmatrix} 3 & 2 & 1 \\ 5 & 3 & 1 \\ 2 & 2 & 8 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 3 \\ 5 & 3 & 1 \\ 2 & 2 & 8 \end{vmatrix}$$

$$= \{3(24 - 2) - 2(40 - 2) + 1(10 - 6)\} + \{2(24 - 2)$$
$$- 3(40 - 2) + 3(10 - 6)\}$$

$$= \{66 - 76 + 4\} + \{44 - 114 + 12\}$$

$$= -6 - 58$$

$$= -64.$$

Here each element of the first row is expressed as the sum of two terms. We have shown that the determinant is expressible as the sum of two determinants we can write the following rule.

Rule V :

If each element of any row or column is the sum of two terms then the determinant can be shown as the sum of two determinants of the same order.

Now let us take the determinants

$$D_1 = \begin{vmatrix} 2 & 3 & 1 \\ 5 & 0 & 6 \\ 7 & 4 & 9 \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} 2 & 5 & 7 \\ 3 & 0 & 4 \\ 1 & 6 & 9 \end{vmatrix}$$

Expanding these two determinants we get,

$$D_1 = 2(0 - 24) - 3(45 - 42) + 1(20 - 0)$$
$$= -48 - 9 + 20$$
$$= -37$$

$$D_2 = 2(0 - 24) - 5(27 - 4) + 7(18 - 0)$$
$$= -48 - 115 + 126$$
$$= -37$$

In this case the second determinant is obtained by interchanging the rows and columns of the first determinant. Thus, we have the following rule.

Rule VI :

If the rows and columns of a determinant are interchanged, the value of the determinant remains the same. (ex. 13)

4.	Use of Determinant in Solving Simultaneous Equations (Cramer's Method) :

(A) For Two Liner Equations :

$a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ are two linear equations of the first degree in x and y. a_1, b_1, c_1, a_2, b_2, c_2 are constants. These equations can be solved by comparing the co–efficients of x or y.

For comparing the co–efficients of y, multiply the first equations by b_2 and the second by b_1.

$$\therefore \quad a_1 b_2 x + b_1 b_2 y + c_1 b_2 = 0$$
$$a_2 b_1 x + b_1 b_2 y + b_1 c_2 = 0$$

$$\overline{(a_1 b_2 - a_2 b_1)x + c_1 b_2 - b_1 c_2 = 0}$$

$$\therefore \qquad x = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}$$

Similarly $\quad y = \dfrac{c_1 a_2 - a_1 c_2}{a_1 b_2 - a_2 b_1}$

$$\therefore \qquad -y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$$

i.e. $\dfrac{x}{b_1 c_2 - b_2 c_1} = \dfrac{1}{a_1 b_2 - a_2 b_1}$

and $\dfrac{-y}{a_1 c_2 - a_2 c_1} = \dfrac{1}{a_1 b_2 - a_2 b_1}$

$$\therefore \quad \frac{x}{b_1 c_2 - b_2 c_1} = \frac{-y}{a_1 c_2 - a_2 c_1} = \frac{1}{a_1 b_2 - a_2 b_1}$$

$$\therefore \quad \frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Thus, we can get the values of x and y. This method of solving simultaneous equations is known as **Cramer's method.**

(B) Cramer's method for solving three simultaneous linear equations

If we have three simultaneous linear equations

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$
$$a_3x + b_3y + c_3z + d_3 = 0$$

We can obtain the solutions as $\dfrac{x}{D_x} = \dfrac{-y}{D_y} = \dfrac{z}{D_z} = \dfrac{-1}{D}$

Where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad D_x = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} \qquad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Then, $x = \dfrac{-D_x}{D},\ y = \dfrac{D_y}{D},\ z = \dfrac{-D_z}{D}$

It should be noted that the equation can be solved only when $D \neq 0$.

Illustration 5 : Solve the equations :

$$3x + 7y + 4 = 0$$
$$4x + y - 3 = 0$$

Ans. : Here $a_1 = 3,\ b_1 = 7,\ c_1 = 4,\ a_2 = 4,\ b_2 = 1,\ c_2 = -3.$

$$\therefore \quad \frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\therefore \quad \frac{x}{\begin{vmatrix} 7 & 4 \\ 1 & -3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & 7 \\ 4 & 1 \end{vmatrix}}$$

$$\therefore \quad \frac{x}{-25} = \frac{-y}{-25} = \frac{1}{-25}$$

$$\therefore \quad x = \frac{-25}{-25} \text{ and } y = \frac{25}{-25}$$

$$\therefore \quad x = 1 \text{ and } y = -1$$

F₉ S.G. B.C.A. Math (seme-1) -8 [50]

Illustration 6 : Solve the equations :

$$2x - 9 = 5y$$
$$x - y = 3$$

Ans. : We shall arrange the equations in proper form.

∴ $\quad 2x - 5y - 9 = 0$

$\quad\quad x - y - 3 = 0$

∴ $\dfrac{x}{\begin{vmatrix} -5 & -9 \\ -1 & -3 \end{vmatrix}} = \dfrac{-y}{\begin{vmatrix} 2 & -9 \\ 1 & -3 \end{vmatrix}} = \dfrac{1}{\begin{vmatrix} 2 & -5 \\ 1 & -1 \end{vmatrix}}$

∴ $\dfrac{x}{15 - 9} = \dfrac{-y}{-6 + 9} = \dfrac{1}{-2 + 5}$

∴ $\dfrac{x}{6} = \dfrac{-y}{3} = \dfrac{1}{3}$

∴ $x = \dfrac{6}{3}; \; -y = \dfrac{3}{3}$

∴ $\quad x = 2; \; y = -1$

Illustration 7 : Solve the equations :

$$y = x + 7$$
$$y = 2x + 15$$

Ans. : Rearranging the equations, we get,

$$-x + y - 7 = 0$$
$$-2x + y - 15 = 0$$

∴ $\dfrac{x}{\begin{vmatrix} 1 & -7 \\ 1 & -15 \end{vmatrix}} = \dfrac{-y}{\begin{vmatrix} -1 & -7 \\ -2 & -15 \end{vmatrix}} = \dfrac{1}{\begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix}}$

∴ $\dfrac{x}{-15 + 7} = \dfrac{-y}{15 - 14} = \dfrac{1}{-1 + 2}$

∴ $\dfrac{x}{-8} = \dfrac{-y}{1} = \dfrac{1}{1}$

$x = -8; \; y = -1$

Illustration 8 : Solve the equations :

$$\frac{x}{3} + \frac{y}{4} = 1$$

$$\frac{2x}{9} - \frac{y}{2} = 6$$

Ans. : Multiplying the 1st equation by 12 and 2nd equation by 18 and rearranging the equations, we get.

$$4x + 3y - 12 = 0$$
$$4x - 9y - 108 = 0$$

$$\therefore \frac{x}{\begin{vmatrix} 3 & -12 \\ -9 & -108 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 4 & -12 \\ 4 & -108 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 4 & 3 \\ 4 & -9 \end{vmatrix}}$$

$$\therefore \frac{x}{-324 - 108} = \frac{-y}{-432 + 48} = \frac{1}{-36 - 12}$$

$$\therefore \frac{x}{-432} = \frac{-y}{-384} = \frac{1}{-48}$$

$$\therefore x = \frac{-432}{-48}; y = \frac{384}{-48}$$

$$\therefore x = 9, \quad y = -8$$

Illustration 9 : Solve the equations :

$$4x + 10y = 2xy$$
$$5x + 16y = 3xy$$

Ans. : Dividing both the equations by xy, we get

$$\frac{4}{y} + \frac{10}{x} = 2, \& \frac{5}{y} + \frac{16}{x} = 3$$

Putting $\frac{1}{x} = a$ and $\frac{1}{y} = b$, we get

$$\therefore 10a + 4b - 2 = 0$$
$$16a + 5b - 3 = 0$$

Now, by Cramer's method,

$$\frac{a}{\begin{vmatrix} 4 & -2 \\ 5 & -3 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 10 & -2 \\ 16 & -3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 10 & 4 \\ 16 & 5 \end{vmatrix}}$$

$$\therefore \frac{a}{-12 + 10} = \frac{-b}{-30 + 32} = \frac{1}{50 - 64}$$

$$\therefore \frac{a}{-2} = \frac{-b}{2} = \frac{1}{-14}$$

$$\therefore a = \frac{2}{14}, b = \frac{2}{14}$$

$$a = \frac{1}{7}, b = \frac{1}{7}$$

$$\therefore \quad \frac{1}{x} = \frac{1}{7}, \frac{1}{y} = \frac{1}{7}$$

$$\therefore \quad x = 7 \;\&\; y = 7$$

Illustration10 : Solve the following equations, using Cramer's rule :
(April, 2010)

$$x + 2y + 3z = 14$$
$$2x + y + z = 7$$
$$5x + 2y + z = 12$$

Ans : The equations can be expressed as

$$x + 2y + 3z - 14 = 0$$
$$2x + y + z - 7 = 0$$
$$5x + 2y + z - 12 = 0$$

$$\therefore \quad \frac{x}{D_x} = \frac{-y}{D_y} = \frac{z}{D_z} = \frac{-1}{D}$$

Where,

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 5 & 2 & 1 \end{vmatrix} \quad = 1(1-2) - 2(2-5) + 3(4-5)$$

$$= -1 + 6 - 3$$
$$= 2$$

$$D_x = \begin{vmatrix} 2 & 3 & -14 \\ 1 & 1 & -7 \\ 2 & 1 & -12 \end{vmatrix} = 2(-12+7) - 3(-12+14) - 14(1-2)$$

$$= -10 - 6 + 14$$
$$= -2$$

$$D_y = \begin{vmatrix} 1 & 3 & -14 \\ 2 & 1 & -7 \\ 5 & 1 & -12 \end{vmatrix} = 1(-12+7) - 3(-24+35) - 14(2-5)$$

$$= -5 - 33 + 42$$
$$= 4$$

$$D_z = \begin{vmatrix} 1 & 2 & -14 \\ 2 & 1 & -7 \\ 5 & 2 & -12 \end{vmatrix} = 1(-12+14) - 2(-24+35) - 14(4-5)$$

$$= 2 - 22 + 14$$
$$= -6$$

$$\therefore \quad \frac{x}{-2} = \frac{-y}{4} = \frac{z}{-6} = \frac{-1}{2}$$

$$\therefore \quad x = 1, y = 2, z = 3$$

Illustration 11 : Solve the equations by Cramer's rule :

$$\frac{3}{x} - \frac{4}{y} - \frac{2}{z} = 1 \; ; \; \frac{1}{x} + \frac{2}{y} + \frac{1}{z} = 2 \; ; \; \frac{2}{x} + \frac{5}{y} - \frac{2}{z} = 3$$

Ans. : Put $\frac{1}{x} = X, \frac{1}{y} = Y, \frac{1}{z} = Z$

∴   The equations can be expressed as

$$3X - 4Y - 2Z - 1 = 0$$
$$X + 2Y + Z - 2 = 0$$
$$2X + 5Y - 2Z - 3 = 0$$

∴ $$\frac{X}{D_X} = \frac{-Y}{D_Y} = \frac{Z}{D_Z} = \frac{-1}{D}$$

Where,

$$D = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & 5 & -2 \end{vmatrix} = 3(-4-5) + 4(-2-2) - 2(5-4)$$

$$= -45$$

$$D_X = \begin{vmatrix} -4 & -2 & -1 \\ 2 & 1 & -2 \\ 5 & -2 & -3 \end{vmatrix} = -4(-3-4) + 2(-6+10) - 1(-4-5)$$

$$= 45$$

$$D_Y = \begin{vmatrix} 3 & -2 & -1 \\ 1 & 1 & -2 \\ 2 & -2 & -3 \end{vmatrix} = 3(-3-4) + 2(-3+4) - 1(-2-2)$$

$$= -15$$

$$D_Z = \begin{vmatrix} 3 & -4 & -1 \\ 1 & 2 & -2 \\ 2 & 5 & -3 \end{vmatrix} = 3(-6+10) + 4(-3+4) - 1(5-4)$$

$$= 15$$

Now, $$\frac{X}{45} = \frac{-Y}{-15} = \frac{Z}{15} = \frac{-1}{-45}$$

∴ $X = 1, Y = \frac{1}{3}, Z = \frac{1}{3}$

∴ $\frac{1}{x} = 1, \frac{1}{y} = \frac{1}{3}, \frac{1}{z} = \frac{1}{3}$

∴ $x = 1, y = 3, z = 3$

Illustration : 12 Evaluate $\begin{vmatrix} 6 & 3 & 9 \\ 1 & 0 & 2 \\ 40 & 50 & 20 \end{vmatrix}$

Ans. : Taking 3 from the first row and 10 from the third row as common factors we get,

$$D = 3 \times 10 \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & 5 & 2 \end{vmatrix}$$

$$= 30[2(0 - 10) - 1(2 - 8) + 3(5 - 0)]$$
$$= 30[-20 + 6 + 15]$$
$$= 30(1)$$
$$= 30$$

Illustration 13 : Evaluate $\begin{vmatrix} 2 & 5 & 7 \\ 6 & 15 & 21 \\ 1300 & 1248 & 1871 \end{vmatrix}$

Ans. :

Taking 3 as common factor from second row we get

$$= 3 \begin{vmatrix} 2 & 5 & 7 \\ 2 & 5 & 7 \\ 1300 & 1248 & 1871 \end{vmatrix}$$

$$= 3 \times 0 \ [\because \text{Two rows are identical.}]$$
$$= 0$$

Illustration 14 : Evaluate $\begin{vmatrix} x+y & z & 1 \\ y+z & x & 1 \\ z+x & y & 1 \end{vmatrix}$

Ans. : $D = \begin{vmatrix} x+y & z & 1 \\ y+z & x & 1 \\ z+x & y & 1 \end{vmatrix}$

Adding the elements of second column to the corresponding elements of first column we get,

$$D = \begin{vmatrix} x+y+z & z & 1 \\ x+y+z & x & 1 \\ x+y+z & y & 1 \end{vmatrix}$$

$$= (x + y + z) \begin{vmatrix} 1 & z & 1 \\ 1 & x & 1 \\ 1 & y & 1 \end{vmatrix}$$

$$= (x + y + z) (0) \ldots \{\because \text{Two columns are identical}\}$$
$$= 0$$

Illustration 15 : Evaluate $\begin{vmatrix} x+1 & 2 & 3 \\ 1 & x+2 & 3 \\ 1 & 2 & x+3 \end{vmatrix}$

Ans. :

$$D = \begin{vmatrix} x+1 & 2 & 3 \\ 1 & x+2 & 3 \\ 1 & 2 & x+3 \end{vmatrix}$$

Adding the elements of second and third columns to the corresponding elements of first column we get,

$$D = \begin{vmatrix} x+6 & 2 & 3 \\ x+6 & x+2 & 3 \\ x+6 & 3 & x+3 \end{vmatrix} = (x+6) \begin{vmatrix} 1 & 2 & 3 \\ 1 & x+2 & 3 \\ 1 & 2 & x+3 \end{vmatrix}$$

Now subtracting the elements of first row from the corresponding elements of second and third rows we get,

$$D = (x + 6) \begin{vmatrix} 1 & 2 & 3 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}$$

Expanding with respect to first column we get

$$D = (x + 6) [1 (x^2 - 0)]$$
$$= x^2 (x + 6)$$

Illustration 16 : Evaluate $\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$

Ans. : $D = \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$

Expanding with respect to first row we get

$$D = x(x^2 - yz) - y(xz - y^2) + z(z^2 - xy)$$
$$= x^3 - xyz - xyz + y^3 + z^3 - xyz$$
$$= x^3 + y^3 + z^3 - 3xyz.$$

Illustration 17 : Evaluate $\begin{vmatrix} 3x + 11 & 3x + 10 & 3x + 8 \\ 3x + 10 & 3x + 9 & 3x + 7 \\ 1965 & 1964 & 1962 \end{vmatrix}$

Ans. :

Here, $D = \begin{vmatrix} 3x+11 & 3x+10 & 3x+8 \\ 3x+10 & 3x+9 & 3x+7 \\ 1965 & 1964 & 1962 \end{vmatrix}$

Subtracting second column from first column and third column from second column we get.

$D = \begin{vmatrix} 1 & 2 & 3x+8 \\ 1 & 2 & 3x+7 \\ 1 & 2 & 1962 \end{vmatrix}$ $C_1 - C_2$
$C_2 - C_3$

$= 2 \times \begin{vmatrix} 1 & 1 & 3x+8 \\ 1 & 1 & 3x+7 \\ 1 & 1 & 1962 \end{vmatrix}$

$= 2(0) ..... (\because \text{Two columns are identical})$

$= 0$

Illustration 18 : Prove that   *(March, April 2009)*

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Ans. :

$\text{L.H.S.} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Subtracting second row from first row, and third row from second row we get.

$= \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$ $R_1 - R_2$
$R_2 - R_3$

$= (a-b)(b-c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$

$= (a-b)(b-c) \{1(b+c-a-b)\} .....$

[Expanding with respect to first column]

$= (a-b)(b-c)(c-a)$

$= \text{R.H.S.}$

Illustration 19: Solve the equation :

$$\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0$$

Ans. :

$$\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0$$

Subtracting second row from the first and third row from second we get,

$$\begin{vmatrix} x-1 & 1-x & 0 \\ 0 & x-1 & 1-x \\ 1 & 1 & x \end{vmatrix} = 0$$

$$\therefore \quad (x-1)(x-1) \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & x \end{vmatrix} = 0$$

$\therefore \quad (x-1)^2 \{1(x+1) + 1(0+1) + 0\} = 0$

$\therefore \quad (x-1)^2 \{ x+1+1\} = 0$

$\therefore \quad (x-1)^2 (x+2) = 0$

$\therefore \quad x-1 = 0 \text{ or } x+2 = 0$

$\therefore \quad x = 1 \text{ or } x = -2$

EXERCISES

Evaluate the following determinants :

1. $\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix}$
[Ans.: 14]

2. $\begin{vmatrix} -6 & 2 \\ -3 & -4 \end{vmatrix}$
[Ans.: 30]

3. $\begin{vmatrix} 7 & 28 \\ 3 & 12 \end{vmatrix}$
[Ans.: 0]

4. $\begin{vmatrix} x+y & x \\ x & x-y \end{vmatrix}$
[Ans.: $-y^2$]

5. $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$
[Ans.: $4ab$]

6. $\begin{vmatrix} 2+a & 2 \\ 3 & 3+a \end{vmatrix}$
[Ans. : $a^2 + 5a$]

Solve the following equations using determinants (Cramer's method)

7. $2x + 3y = 13, x + y = 5$ [Ans. : $x = 2, y = 3$]

8. $y = 1 - 3x, 2y = 4 - 5x$ [Ans. : $x = -2, y = 7$]

9. $8x + 3y - 13 = 0, 7x + 5y - 9 = 0$ [Ans. : $x = 2, y = -1$]

10. $2x + 3y + 5 = 0, 3x + 4y + 7 = 0$ [Ans. : $x = -1, y = -1$]

11. $2x - 3y = 3, 5x - 7y = 2$ [Ans. : $x = -15, y = -11$]

12. $2y = 3 - x, 5y = 7 - 4x$ [Ans. : $x = -\dfrac{1}{3}, y = \dfrac{5}{3}$]

13. $\dfrac{x}{3} + \dfrac{7y}{3} = 9, \dfrac{3x}{2} - \dfrac{y}{3} - 8 = 0$ [Ans. : $x = 6, y = 3$]

14. $\dfrac{1}{x} - \dfrac{2}{y} = 6, \dfrac{3}{x} + \dfrac{5}{y} = 7$ [Ans. : $x = \dfrac{1}{4}, y = -1$]

15. $3x - 4y = 24, 3y + 2x + 1 = 0$ [Ans. : $x = 4, y = -3$]

16. $4(x - 1) + 3(y - 1) = 15, 3(x - 1) + 4(y + 1) = 21$ [Ans. : $x = 4, y = 2$]

17. $2x + 5y = 2xy, 4x + 5y = 3xy$ [Ans. : $x = 5, y = 2$]

18. $2x - 6y = 5xy, 6x - 5y = 2xy$ [Ans. : $x = -1, y = -2$]

19. $\begin{vmatrix} x+2 & 3 \\ y+1 & 5 \end{vmatrix} = 8, \begin{vmatrix} x-1 & y-1 \\ 1 & 6 \end{vmatrix} = 4$ [Ans.: $x = 2, y = 3$]

20. $\begin{vmatrix} x+2 & 2y \\ 1 & 3 \end{vmatrix} = 25, \begin{vmatrix} 3 & -4 \\ x-2 & y \end{vmatrix} = 23$ [Ans. : $x = 7, y = 1$]

21. Solve the equations : (use Cramer's rule)

(Nov., Dec 2008; March, April 2009)

$$3x + 5y + 6z = 4$$
$$x + 2y + 3z = 2$$
$$2x + 4y + 5z = 3$$

[Ans. : $x = 1, y = -1, z = 1$]

22. Solve the equations : (use Cramer's rule)

$$x + y = -1$$
$$y + z = 1$$
$$z + x = 0$$

[Ans. : $x = -1, y = 0, z = 1$]

23. Solve the equations : (use Cramer's rule)

$$2x + 3y - z = 5$$
$$3x + 2y + z = 10$$
$$x - 5y + 3z = 0$$

[Ans. : $x = 1, y = 2, z = 3$]

24. Evaluate $\begin{vmatrix} 4 & 45 & 55 \\ 2 & 29 & 32 \\ 6 & 68 & 87 \end{vmatrix}$ [Ans. : 108]

Find the values of the following determinants :

25. $\begin{vmatrix} 2 & 3 & 4 \\ 2 & 7 & -7 \\ 4 & 9 & 1 \end{vmatrix}$ [Ans. : 10]

26. $\begin{vmatrix} 2 & -3 & -3 \\ 5 & -7 & -2 \\ 7 & -8 & 17 \end{vmatrix}$ [Ans. : 0]

27. $\begin{vmatrix} x & 1 & 2 \\ 2 & 2 & -x \\ 3 & x & 4 \end{vmatrix}$ [Ans. : $x^3 + 9x - 20$]

28. $\begin{vmatrix} 1 & -2 & -7 \\ 2 & -3 & -12 \\ -3 & 4 & 17 \end{vmatrix}$ [Ans. : 0]

29. Find the values of (i) $\begin{vmatrix} 5 & -4 & -12 \\ 0 & 3 & -5 \\ 0 & 2 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 5 & 2 & 3 \\ 12 & 25 & 50 \\ 5 & 2 & 3 \end{vmatrix}$

[Ans. : (i) 15 (ii) 0]

30. If $\begin{vmatrix} 11 & 40 & 28 \\ 3 & 12 & 8 \\ A & 2 & 2 \end{vmatrix} = 0$, Find the value of A.   [Ans. : A = 1]

31. If $\begin{vmatrix} 16 & 8 & 26 \\ 6 & 3 & 9 \\ 2 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 2 & K & 0 \\ 7 & 14 & 9 \end{vmatrix}$ Find the value of K. [Ans. : K = 4]

32. Find the value of K if $\begin{vmatrix} 4 & 5 & -7 \\ -2 & K & 6 \\ 1 & K & 1 \end{vmatrix} = 43$ [Ans. : K = 3]

Solve the following equations :

33. $\begin{vmatrix} 1 & 2 & -3 \\ x & 2 & -5 \\ 4 & -3 & -1 \end{vmatrix} = 0$ [Ans. : $x = 3$]

34. $\begin{vmatrix} 3 & x & -8 \\ 5 & -2 & -3 \\ 1 & -3 & 2 \end{vmatrix} = 0$ [Ans. : $x = 5$]

35. Solve : $\begin{vmatrix} x & y & 1 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix} = 0$ and $\begin{vmatrix} x & y & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 8 \end{vmatrix} = 0$

[Ans. : $x = \dfrac{11}{5}$, $y = \dfrac{8}{5}$]

36. Find the value of $\begin{vmatrix} 201 & 210 & 220 \\ 151 & 155 & 140 \\ 50 & 55 & 80 \end{vmatrix}$ **[Ans. : 0]**

37. Find the value of $\begin{vmatrix} 100 & 205 & 105 \\ 200 & 408 & 207 \\ 300 & 608 & 310 \end{vmatrix}$ **[Ans. : $-$ 1100]**

38. Prove that $\begin{vmatrix} 1977 & 1979 & 1981 \\ 1940 & 1943 & 1946 \\ 10 & 17 & 24 \end{vmatrix} = 0$

39. Prove that $\begin{vmatrix} 2 & 1970 & 1978 \\ 5 & 1960 & 1980 \\ 7 & 1950 & 1978 \end{vmatrix} = 0$

Simplify :

40. $\begin{vmatrix} x+2 & x+5 & x+8 \\ 37 & 40 & 43 \\ 1971 & 1974 & 1977 \end{vmatrix}$ **[Ans. : 0]**

41. Solve by Cramer's method
$x + 6y = 2xy$
$3x + 2y = 2xy$ **[Ans. : $x = 4$, $y = 2$]**

42. Solve using Cramer's method
$\dfrac{9}{x} + \dfrac{2}{y} = 4;\ \dfrac{15}{x} - \dfrac{4}{y} = 3$ **[Ans. : $x = 3$, $y = 2$]**

43. $ax + by - ab = 0$
$bx + ay - ab = 0$

[Ans.: $x = \dfrac{ab}{a+b}, y = \dfrac{ab}{a+b}$]

44. Solve the equations using Cramer's rule :
$2x - 3y + z = 3,$
$x + y - 2z = -1,$
$3x - 2y + 2z = 8$

45. Solve the equations using determinants :   **[Ans. : $x = 2$, $y = 1$, $z = 2$]**
$x + 2y = 3,$
$y - 3z = 4,$
$3x - 2z = 5$

[Ans. : $x = 1$, $y = 1$, $z = -1$]

University Questions

1. Answer the following : (April, 2010)

(i) Find the value of $\begin{vmatrix} a & a-b \\ b & b-a \end{vmatrix}$

[Ans. : $b^2 - a^2$]

(ii) Solve the system of equation by Cramer's method :
$x + 2y + 3z = 14,$
$2x + y + z = 7$ and
$5x + 2y + z = 12$

(Also in Nov./Dec. 2008)
[Ans. : $x = 1,\ y = 2,\ z = 3$]

(iii) Prove that :

$$\begin{vmatrix} y+z & z+x & x+y \\ x+y & y+z & z+x \\ z+x & x+y & y+z \end{vmatrix} = 2 \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$$

(iv) Find the solution of system of equations :
$x + 2y = 3,\ 2x + 7y = 9.$ [Ans. : $x = 1,\ y = 1$]

2. Answer the following : (March/April, 2009)

(i) Solve the following using Cramer's rule :
$x + 2y - z = 2$
$3x + 6y + z = 1$ and

$3x + 3y + 2z = 3$ [Ans. : $x = \dfrac{35}{12},\ y = \dfrac{-13}{12},\ z = \dfrac{-5}{4}$]

(ii) Prove that

$$A = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

(iii) Solve, using Cramer's rule :
$3x + 5y + 6z = 4,$
$x + 2y + 3z = 2$ and
$2x + 4y + 5z = 3.$

(in Sept/Oct. 2009,)
(Also in Nov./Dec. 2008)
[Ans. : $x = 1,\ y = -1,\ z = 1$]

(iv) Prove that

$$\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

(Also in March / April 2007)
(Oct./Nov, 2009)

3. Answer the following :

(i) Evaluate $\begin{vmatrix} x+y & -y \\ y & x-y \end{vmatrix}$ [Ans. : x^2]

(ii) Solve the following System of equations :

$x + y + z = 3$

$2x + y + z = 4$

$2x + 2y + 9z = 13$, by Cramer's rule.

[Ans. : $x = 1, y = 1, z = 1$]

(iii) Prove that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x - y)\,(y - z)\,(z - x)$

(iv) Solve the given System of equations by Cramer's rule.

$x + 2y + 3z = 5,\ 2x + 3y + 4z = 8,\ 3x + 5y + 6z = 10$

[Ans. : $x = 4, y = -4, z = 3$]

4. **Answer the following :**

(i) Prove that

$$\begin{vmatrix} (a-1)^2 & (b-1)^2 & (c-1)^2 \\ 1 & 1 & 1 \\ a+1 & b+1 & c+1 \end{vmatrix} = (a-b)\,(b-c)\,(c-a)$$

(Sept./Oct. 2009)

(ii) What is the difference between minor and confactor of the element of determinant ?

Explain with illustration. *(Nov./Dec. 2008)*

(iii) Solve the equations $2x + y - 8 = 0,\ x + 2y - 1 = 0$, using Cramer's rule. *(March/April 2007)*

[Ans. : $x = 5, y = -2$]

(iv) Solve the equations $2x + 3y = 13,\ x + y = 5$, using Cramer's rule. *(March/April 2007)*

[Ans. : $x = 2, y = 3$]

(v) Solve using Cramer's rule :

$2x - 3y + z = 3$

$x + y - 2z = 1$

$3x - 2y + 2z = 8$ *(Dec. 2007)*

(vi) If $\begin{vmatrix} 11 & 40 & 28 \\ 3 & 12 & 8 \\ A & 2 & 2 \end{vmatrix} = 0$ then find the value of A. *(Dec. 2007)*

[Ans. : $A = 1$]

(vii) Obtain $\begin{vmatrix} x+y & x \\ x & x-y \end{vmatrix}$ *(Dec. 2007)*

[Ans. : $-y^2$]

■ ■ ■